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An Alternative Approach for Solving Unbalanced Assignment Problems

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Abstract

This paper is devoted to present a new approach to make an unbalanced assignment problem into a balanced one and a comparison is carried out with the existing methods. The proposed approach is quite simple and different from the existing approach available in the literature. Numerical examples are provided to demonstrate the efficiency and validity of the proposed approach. The results of computational experiments on a large number of test problems reveal that the proposed method is capable of assigning a number of tasks to an equal number of agents optimally.

Keywords: Assignment problem, Balanced assignment problem, Unbalanced assignment problem, Hungarian method, Optimal solution.

Introduction

The assignment problem is a combinatorial optimization problem in the field of operations research. It is a special case and completely degenerate form of a transportation problem, which occurs when each supply is 1 and each demand is 1. It consists of assigning a number of tasks to an equal number of agents (each agent is assigned to exactly one task and each task has exactly one agent assigned to perform it) in such a way that the objective is either to minimize the total cost or to maximize the total profit. Here agents and tasks are treated as sources and destinations respectively. The agents could be people, machines, vehicles, plants etc. It is applicable in assigning personnel to offices, machines to jobs, vehicles to routes, salesman to sales territories, teachers to classes, aircrafts to particular trips, products to factories and so on. Because of being a special case of transportation problem, the assignment problem can be solved by transportation technique (North-West corner rule, Least cost method, Vogel's approximation method etc.) or simplex method. The most popular and basic combinatorial method for assignment problem is the Hungarian method. The method was developed and published by H.W. Kuhn [1]. He

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gave the name Hungarian method, because the algorithm was largely based on the earlier works of two Hungarian mathematicians: D. Konig and J. Egervary. James Munkres [2] reviewed the algorithm and observed that it is strongly polynomial. Since then the algorithm has been known as Kuhn-Munkres algorithm or Munkres assignment algorithm. Besides the Hungarian method, the simplex method for linear programming was modified to solve the assignment problem [3],[4],[5]. Also the signature method for the assignment problem was presented by Balinski [6]. Kore [7] proposed a new approach to solve an unbalanced assignment problem without balancing it. Basirzadeh [8] developed a Hungarian-like method, called O*nes Assignment Method* which can be applied for assignment problems. This method is based on generating some ones instead of zeros in the assignment matrix and an assignment is made in terms of the ones. Ghadle and Muley [9] introduced *Revised Ones Assignment Method* by taking a simple modification on *Ones Assignment Method*. Gamal [10] identified some drawbacks to *Ones Assignment Method* and provide a remedial strategy to overcome a case when the entries of the cost matrix are zeros. Rupsha Roy and Rukmani Rathore [11] applied zero suffix method [12] for finding an optimal solution of an assignment problem. Abdur Rashid[13] investigated several heuristic algorithms for solving cost minimizing and profit maximizing assignment problems. In some cases in the assignment matrix, the number of agents are not equal to the number of tasks. Such problems can be termed as unbalanced assignment problem. The effectiveness matrix in this case will not be a square matrix. In such problems, dummy row(s) or dummy column(s) are added in the matrix so as to complete it to form a square matrix. In traditional approach, costs in such rows or columns represented by dummy will be treated as **zero**. Having formed to balanced matrix, the problem is then solved by Hungarian method. In literature a good amount of research [14,15,16,17,18] is available for solving unbalanced assignment problems. In this paper we propose an efficient approach to form an unbalanced assignment problem into a balanced one and then obtain an optimal solution in an assignment problem of unbalanced type.

Mathematical Formulation of Assignment Problem

As the assignment problem is a particular case of the transportation problem, it can be formulated as a linear programming problem (LPP). Suppose there are n tasks to be performed by n agents. Each task must be assigned to one and only one agent and each agent has to complete one and only one task (assignments are made on a one-to-one basis). Let c_{ij} be the cost of assigning agent i to task j and also let

$$x_{ij} = \begin{cases} 1, \text{ if agent } i \text{ is assigned to task } j \\ 0, \text{ if agent } i \text{ is not assigned to task } j \end{cases}$$

,the problem is to find an assignment (which task should be assigned to which agent) so that the total cost for performing all tasks is minimum. Thus the mathematical formulation of the standard assignment problem is as follows:

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,......, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,2,......, n$$

$$x_{ij} \in \{1, 0\}$$

The first set of constraints ensures that each task is assigned to only one agent and the second set of constraints ensures that each agent is assigned to only one task.

From the above formulation, it is pointed out that the assignment problem is mathematically a 0-1 integer programming problem. The assignment problem can be stated in the form of $n \times n$ effectiveness matrix [c_{ij}] as shown in the following.

General Assignment Matrix

	1	2	…	n	Supply(a_i)
1	c_{11}	c_{12}	…	c_{1n}	1
2	c_{21}	c_{22}	…	c_{2n}	1
⋮	⋮	⋮	⋮	⋮	⋮
n	c_{n1}	c_{n2}	…	c_{nn}	1
Demand(b_j)	1	1	…	1	

Hence it is remarkable that an assignment problem is an $n \times n$ transportation problem with each $a_i = b_j = 1$, i.e. there would be only one assignment in a given row or a column of the effectiveness matrix.

Hungarian method

The algorithm of the Hungarian method consists of the following steps:

Step1: *Determine the Total opportunity cost matrix*.

(i) Subtract the minimum element in each row of the given payoff matrix from all the elements of the respective rows.

(ii) Subtract the minimum element in each column of the matrix obtained in step 1(i) from all the elements of the respective columns.

Hence a total opportunity cost matrix is obtained.

Step2: *Determine whether an optimal assignment can be obtained.*

(i)Draw the minimum number of horizontal and vertical lines to cover all the zeros in the total opportunity cost matrix.

(ii)If the number of lines drawn in step 2(i) is equal to the order of the payoff matrix then an optimal assignment can be made as usual procedure so that the total opportunity cost involved in the assignment is zero.

(iii)If the number of lines drawn in step 2(i) is less than the order of the payoff matrix, go to step 3.

Step3: *Revise the total opportunity cost matrix.*

(i) Identify the smallest uncovered element in the current total opportunity cost matrix.
(ii) Subtract this smallest element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines.
Thus a revised total opportunity cost matrix is obtained.

Step4: Repeat steps 2 and 3 until the step 2(ii) is attained.

Proposed Method

In this section, we present an effective approach to make an unbalanced assignment problem into a balanced one. The basic principle on which the procedure of our approach is based can be stated as the following theorem.

Theorem: If a constant is added to any row or column in the assignment cost matrix [c_{ij}], then the optimal solution remains the same.

Now the proposed algorithm consists of the following steps:

Step 1: Balance the given assignment problem if either (number of row(s)$>$ number of column(s)) or (number of row(s)$<$ number of column(s)) by introducing dummy column(s) or dummy row(s) respectively. Costs in such rows or columns represented by dummy will be regarded as any **constant element**.

In traditional approach, this is done by introducing zero unit costs.

Step 2: If dummy row(s) is/are added, make row reduction only by subtracting the smallest element in each row from each element in that row. Again if dummy column(s) is/are added, make column reduction only by subtracting the smallest element in each column from each element in that column. In this way at least one zero is obtained in each row and each column.

Step 3: Make zero assignments following the procedure of Hungarian method [1].

Step 4: Compute total cost corresponding to the assignments (ignoring assignments in dummy row or column) obtained in step 3 using the original cost matrix.

Numerical Illustration

Example 1: There are four jobs to be assigned to five machines. Only one job can be assigned to one machine. The amount of time in hours required for the jobs per machine are given in the following matrix.

		Machines				
		A	B	C	D	E
Jobs	1	4	3	6	2	7
	2	10	12	11	14	16
	3	4	3	2	1	5
	4	8	7	6	9	6

It is required to find an optimum assignment of jobs to the machines to minimize the total processing time.

In this problem, the number of rows is less than the number of columns. Adding one dummy row with constant element, the balanced matrix is obtained as follows.

		Machines				
		A	B	C	D	E
Jobs	1	4	3	6	2	7
	2	10	12	11	14	16
	3	4	3	2	1	5
	4	8	7	6	9	6
	5	220	220	220	220	220

Now making row reduction only, we obtain the following reduced matrix.

		Machines				
		A	B	C	D	E
Jobs	1	2	1	4	0	5
	2	0	2	1	4	6
	3	3	2	1	0	4
	4	2	1	0	3	0
	5	0	0	0	0	0

Applying Hungarian method the optimal solution is obtained as follows.

Iteration 1:

Machines

Jobs	A	B	C	D	E	
1	2	1	4	[0]	5	√
2	[0]	2	1	4	6	
3	3	2	1	~~0~~	4	√
4	2	1	[0]	3	~~0~~	
5	~~0~~	[0]	~~0~~	~~0~~	~~0~~	
				√		

Iteration 2:

Machines

Jobs	A	B	C	D	E
1	1	~~0~~	3	[0]	4
2	[0]	2	1	5	6
3	2	1	[0]	~~0~~	3
4	2	1	~~0~~	4	[0]
5	~~0~~	[0]	~~0~~	1	~~0~~

Hence the optimum assignment schedule is

$1 \rightarrow D,\ 2 \rightarrow A,\ 3 \rightarrow C,\ 4 \rightarrow E$

Since job 5 is dummy, machine B is assigned no job and the corresponding minimum time is $(2+10+2+6)=20$ Hours.

It is interesting that after applying transportation algorithm or simplex method (checked by TORA optimization software) on the given problem, the same optimal solution is also obtained.

Example 2: Consider the following typical unbalanced assignment problem:

	1	2	3	4
A	3	6	2	6
B	7	1	4	4
C	3	8	5	8
D	6	4	3	7
E	5	2	4	4
F	5	7	6	2

Here the number of columns is less than the number of rows. Adding two dummy columns with **constant cost elements** we obtain the balanced cost matrix as follows:

	1	2	3	4	5	6
A	3	6	2	6	100	85
B	7	1	4	4	100	85
C	3	8	5	8	100	85
D	6	4	3	7	100	85
E	5	2	4	4	100	85
F	5	7	6	2	100	85

Now making column reduction only, we obtain the following reduced matrix:

	1	2	3	4	5	6
A	0	5	0	4	0	0
B	4	0	2	2	0	0
C	0	7	3	6	0	0
D	3	3	1	5	0	0
E	2	1	2	2	0	0
F	2	6	4	0	0	0

Applying Hungarian method, the optimal solution is shown below:

	1	2	3	4	5	6
A	0	5	[0]	4	0	0
B	4	[0]	2	2	0	0
C	[0]	7	3	6	0	0
D	3	3	1	5	[0]	0
E	2	1	2	2	0	[0]
F	2	6	4	[0]	0	0

Hence the optimum assignment schedule is

$A \rightarrow 3, \; B \rightarrow 2, C \rightarrow 1, \; F \rightarrow 4$

where D and E are remained free from any assignment and the corresponding minimum cost is $(2+1+3+2)=8$ unit.

It is noted that the same optimal solution is also obtained by employing transportation technique or simplex method on the given problem (checked by TORA optimization software).

Comparative Study

Here we carry out a comparison of the proposed method with the other existing methods available in the literature. Our study in this regard is summarized in the table below.

Table 1: A Comparative study of different methods

Examples	Optimum Value	Number of iterations required using			
		Proposed method	North-West Corner Method	Least Cost Method	Vogel's Approximation Method
1	20	2	10	6	5
2	8	1	13	9	2

This table clearly shows that the proposed method requires fewer number of iterations, than the other three, to get the optimal solution.

Conclusion

In this paper we have provided a different approach for finding optimal solution of an unbalanced assignment problem. The solution procedure is illustrated with the help of numerical example and verified with TORA optimization software. The methodology proposed herein is very simple, easy to understand and apply. Thus the proposed method can help decision makers in handling assignment problems which are of unbalanced type occurring in real life situations and providing an optimal solution in a simple and effective manner.

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