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Inference for the Scale Parameter of the Class of Life-Time Distributions Using Transformed Chi-square Family

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Abstract

In this paper, an attempt has been made to obtain inference for the scale parameter of the class of life-time distributions using a family of transformed Chi-square distributions, a sub-family of the exponential family of probability distributions having special characteristics.

Keywords: Chi-square, distribution, sub-family, statistic, complete, unbiased and estimator.

Introduction

The class of life-time distributions introduced by Prakash and Singh [5] is an important life-time distribution in survival analysis. Suppose a random variable X follows the distribution presented by a class of probability density functions (p.d.f) with scale parameter θ and two known positive constants p and q is given as

$$f(x;\theta)=\frac{q}{\Gamma p}\frac{1}{\theta^{p}}x^{pq-1}e^{-\frac{1}{\theta}x^{q}} \ ; \ x\geq 0,\ \theta>0,\ p>0,\ q>0\,. \quad (1)$$

The continuous distributions such as negative exponential, gamma, Erlang, Weibull, Rayleigh and Maxwell are particular form of the model (1). The following are the relations between the above distribution for different values of p and q.

(a) When $p=1$ and $q=1$, the model (1) and negative exponential distribution are identical.
(b) When $p=p$ and $q=1$, it reduces to two- parameter gamma distribution.

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(c) When $p =$ positive integer and $q = 1$, it reduces to Erlang distribution.
(d) When $p = 1$ and $q = q$, it converges to Weibull distribution.
(e) When $p = 1$ and $q = 2$, it reduces to Rayleigh distribution.
(f) When $p = \frac{3}{2}$ and $q = 2$, it is same as Maxwell distribution.

In this paper, we have first showed that the class of life-time distributions (1) belongs to an exponential family of distributions and hence a family of transformed Chi-square distributions introduced by Rahman and Gupta [6]. Then an attempt has been made to obtain inference such as sufficient statistic, complete minimal sufficient statistic, maximum likelihood estimator (MLE), minimum variance bound (MVB), minimum variance bound estimator (MVBE), uniformly minimum variance unbiased estimator (UMVUE), Pitman estimator, $100(1-\alpha)\%$ confidence interval and one-sided uniformly most powerful (UMP) test, for the scale parameter of the class of life-time distributions by using a family of transformed Chi-square distributions.

However, it reveals that MLE, MVBE and UMVUE for scale parameter are same and unbiased estimator with minimum variance, where as the Pitman estimator is biased. It has also been seen that the distribution (1) has a monotonic likelihood ratio (MLR) and related to gamma distribution, hence a central Chi-square distribution.

Family of Transformed Chi-Square Distributions

Suppose $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed (i.i.d) random variables from a one-parameter exponential family introduced by Barndorff-Nelsen [7] and $f(x;\theta)$ is given by

$$f(x:\theta) = \exp[b(\theta)a(x) + c(\theta) + h(x)] \tag{2}$$

Rahman and Gupta [6] proved the following Theorem for the family of distributions.

Theorem 2.1 : In a family of distributions (2), the function $-2b(\theta)a(X)$ has Gamma $\left(\frac{k}{2}, \frac{1}{2}\right)$ distribution if and only if

$$\frac{2c'(\theta)b(\theta)}{b'(\theta)} = k\,, \tag{3}$$

where k is positive and free from θ. In case of k is an integer $-2b(\theta)a(X)$ follows a central Chi-square distribution with k degrees of freedom.

Definition 2.1 The one-parameter exponential family of form (2), satisfying (3) is called the family of transformed Chi-square distributions, provided k is a positive integer.

Using (3), Mahmoudi [3] expressed the family of distributions (2) in such as reduced form

$$f(x;\eta) = c(x)\eta^{\nu} e^{-\eta\, a(x)}, \tag{4}$$

where $c(x) = e^{h(x)+k_1}$, $\eta = -b(\theta) > 0$ and $\nu = \frac{k}{2} > 0$. Also note that $-b(\theta)a(X) = 2\eta a(X)$ has Gamma $\left(\nu, \frac{1}{2}\right)$ distribution or $a(X) \approx$ Gamma $\left(\nu, \frac{1}{\eta}\right)$.

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from a distribution that belongs to the family of transformed Chi-square having density of the form (2) and satisfying (3), then the following criteria are used to obtain inference for the scale parameter θ of the class of life-time distributions. Basically, these criteria are a time saving device in estimation and tests of hypotheses that one can easily apply without a vast knowledge but a little knowledge in mathematical statistics and inference.

(i) $-2b(\theta)a(X)$ is distributed as Chi-square with k degrees of freedom.

(ii) Similarly, $-2b(\theta)\sum a(X)$ is distributed as Chi-square with nk degrees of freedom.

(iii) $\sum a(X)$ is a sufficient statistic for θ.

(iv) $\sum a(X)$ is a complete sufficient statistic for θ.

(v) $\dfrac{\sum a(X)}{n}$ is the maximum likelihood estimator for $\left[-\dfrac{k}{2b(\theta)}\right]$.

(vi) $\dfrac{\sum a(X)}{n}$ is a uniformly minimum variance unbiased estimator (UMVUE) for $\left[-\dfrac{k}{2b(\theta)}\right]$.

with minimum variance $\left[\dfrac{k}{2nb^2(\theta)}\right]$.

(vii) $\left[\dfrac{\chi^2_{nk,\alpha}}{2\sum a(X)}, \dfrac{\chi^2_{nk,(1-\alpha)}}{2\sum a(X)}\right]$ is a $100\{1-(\alpha_1+\alpha_2)\}\%$ confidence interval of $[-b(\theta)]$.

(viii) It has a monotonic likelihood ratio (MLR) in $\sum a(X)$.

(ix) An α level one-sided uniformly most powerful (UMP) test for testing the hypothesis

$H_0 : \theta \le \theta_0$ against $H_0 : \theta > \theta_0$ is $\varphi(\underset{-}{X}) = 1$ if $\sum a(X) \ge \dfrac{\chi^2_{nk,\,(1-\alpha)}}{-2b(\theta_0)}$,

provided $b(\theta)$ is

strictly increasing in θ. Also the power function of the test is

$$P_0\left[\chi^2_{nk} \ge \frac{b(\theta)\chi^2_{nk,(1-\alpha)}}{b(\theta_0)}\right].$$

Main Results

In this section, we state the results as a theorem and then prove them on the basis of the mentioned family in Section 2.

Theorem 3.1 The class of life-time distributions model belongs to a one-parameter exponential

family of distributions.

Proof: Let X be a random variable having probability density function (1) given as

$$f(x;\theta)=\frac{q}{\Gamma p}\frac{1}{\theta^{p}}x^{pq-1}e^{-\frac{1}{\theta}x^{q}} \quad ; \; x>0,\ \theta>0,\ p,q>0\,.$$

$$=\exp\left[-\frac{1}{\theta}x^{q}-p\log\theta+\log\left(\frac{q}{\Gamma p}x^{pq-1}\right)\right]$$

$$=\exp[b(\theta)a(X)+c(\theta)+h(X)], \tag{5}$$

where $a(X)=X^{q}$, $b(\theta)=-\frac{1}{\theta}$, $c(\theta)=-p\log\theta$ and $h(X)=\log\left(\frac{q}{\Gamma p}X^{pq-1}\right)$

Since the model (1) satisfies (2) clearly, the class of life-time distributions belongs to the one-parameter exponential family of distributions.

Theorem 3.2 The class of life-time distributions model also belongs to the transformed Chi-square family of distributions.

Proof: Suppose X be a random variable with p.d.f in (1) and re-written as

$$f(x;\theta)=\exp[b(\theta)a(X)+c(\theta)+h(X)],$$

where $a(X)$, $b(\theta)$, $c(\theta)$and $h(X)$are defined above.

Here, $\frac{2c'(\theta)b(\theta)}{b'(\theta)}=k$, where $k=2p$, $p>0$ is positive and free from θ, then $-2b(\theta)a(X)$

follows a gamma distribution with parameters $\frac{k}{2}$ and $\frac{1}{2}$. In case of k is an integer then $-2b(\theta)a(X)$ follows a central Chi-square distribution with k degrees of freedom.

Since the model (1) satisfies (2) and (3), therefore according to the **Definition 2.1** in **Section 2**, the class of life-time distributions belongs to the family of transformed Chi-square distributions.

Theorem 3.3 Let X be a random variable distributed as the class of the life-time distributions (1),

then

(a) $\sum X^q$ is distributed as gamma with parameters np and $\frac{1}{\theta}$.

(b) $\frac{1}{np}\sum X^q$ is distributed as gamma with parameters np and $\frac{np}{\theta}$.

(c) $\frac{2}{\theta}\sum X^q$ is distributed as central Chi-square with $2np$ degrees of freedom.

Proof: (a) Let X be a random variable which follows the p.d.f (1) and satisfies (2) and (3). Also let us assume that $u = x^q$ then (1) becomes

$$f(u;\theta) = \frac{1}{\Gamma p}\frac{1}{\theta^p} e^{-\frac{1}{\theta}u} u^{p-1}; \quad u > 0,\ \theta > 0,\ p > 0, \qquad (6)$$

which is distributed as $u \sim Gamma\left(p, \frac{1}{\theta}\right)$.

Hence, $\sum u = \sum a(X) = \sum x^q$ is also distributed as gamma with parameters np and $\frac{1}{\theta}$.

Therefore, $\sum X^q$ is distributed as gamma with parameters np and $\frac{1}{\theta}$.

(b) Again, let us assume that $u_1 = \frac{1}{np} x^q$ in (1) and it becomes

$$f(u_1;\theta) = \frac{\left(\frac{np}{\theta}\right)^p}{\Gamma p} e^{-\frac{np}{\theta}u_1} u_1^{\,p-1}; \quad u_1 > 0,\ \theta > 0,\ p > 0, \qquad (7)$$

which is distributed as $u_1 \sim Gamma\left(p, \frac{np}{\theta}\right)$.

Hence, $\sum u_1 = \frac{1}{np}\sum a(X) = \frac{1}{np}\sum x^q$ is also distributed as gamma with parameters np and $\frac{np}{\theta}$.

Therefore, $\frac{1}{np}\sum X^q$ is distributed as gamma with parameters np and $\frac{np}{\theta}$.

(c) Now let us consider $u_2 = \frac{2}{\theta} x^q$, then (1) gives

$$f(u_2;\theta) = \frac{\left(\frac{1}{2}\right)^p}{\Gamma p} e^{-\frac{1}{2}u_2} u_2^{\,p-1}; \quad u_2 > 0,\ p > 0, \qquad (8)$$

which is distributed as $u_2 \sim Gamma\left(p, \frac{1}{2}\right)$.

And $\sum u_2 = \frac{2}{\theta}\sum x^q$ is distributed as gamma with parameters np and $\frac{1}{2}$.

Hence $\sum u_2 = -2b(\theta)\sum a(X) = \frac{2}{\theta}\sum x^q$ is distributed as central Chi-square with $2np$

degrees of freedom.

Therefore, $\frac{2}{\theta}\sum X^{q}$ is distributed as central Chi-square with $2np$ degrees of freedom.

Theorem 3.4 $\sum X^{q}$ is a sufficient statistic for the scale parameter θ of the class of life-time distributions.

Proof: Let $\underset{-}{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from the population having p.d.f (1). Then the likelihood function is defined as

$$L(\theta \mid \underset{-}{X}) = \exp\left[-\frac{1}{\theta}\sum X^{q} - np\log\theta + \log\left\{\left(\frac{q}{\Gamma p}\right)^{n}\prod X^{pq-1}\right\}\right]$$

$$= \exp\left[-\frac{1}{\theta}\sum X^{q} - np\log\theta + \left\{n\log\left(\frac{q}{\Gamma p}\right) + (pq-1)\sum\log X\right\}\right]$$

$$= \exp\left[b(\theta)\sum a(X) + nc(\theta) + \sum h(X)\right], \qquad (9)$$

where $a(X) = X^{q}$, $b(\theta) = -\frac{1}{\theta}$, $c(\theta) = -p\log\theta$ and $h(X) = \log\left(\frac{q}{\Gamma p}X^{pq-1}\right)$

According to the Neymann-Fisher Factorization Theorem
$\sum a(X) = \sum X^{q}$ is a sufficient statistic for θ.

Therefore, $\sum X^{q}$ is a sufficient statistic for the scale parameter θ of the class of the life-time distributions.

Theorem 3.5 $\sum X^{q}$ is a complete minimal sufficient statistic for θ of the class of life-time distributions.

Proof: Suppose $\underset{-}{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from the class of life-time distributions having p.d.f. in (1). Then using Theorem 3.4 $T(\mathbf{x}) = \sum a(X) = \sum X^q$ is a sufficient statistic for θ and by Theorem 3.3 (a), and Theorem 6.2.13 given in [1], $\sum X^q$ is a complete minimal sufficient statistic for θ of the class of life-time distributions.

Theorem 3.6 $\frac{1}{np}\sum X^q$ is a maximum likelihood estimator for the scale parameter θ of the class of life-time distributions.

Proof: Let $\underset{-}{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from the population having p.d.f (1) which satisfies (2) and (3). The likelihood function in (9) as

$$L(\theta \mid \underset{-}{X}) = \exp\left[b(\theta)\sum a(X) + nc(\theta) + \sum h(X)\right].$$

Then the log-likelihood function of θ given the sample as

$$\ln L(\theta \mid \underset{-}{X}) = b(\theta)\sum a(X) + nc(\theta) + \sum h(X), \qquad (10)$$

where $a(X)$, $b(\theta)$, $c(\theta)$ and $h(X)$ are mentioned earlier.

Now, differentiating (10) partially with respect to θ and equating to zero, it gives

$$-\frac{k}{2b(\theta)} = \frac{\sum a(X)}{n}.$$

Putting the values of $k = 2p$, $\sum a(X) = \sum X^q$ and $b(\theta) = -\frac{1}{\theta}$, we get

$$\hat{\theta} = \frac{1}{np}\sum X^q$$

Therefore, $\frac{1}{np}\sum X^q$ is a maximum likelihood estimator for the scale parameter θ of the class of life-time distributions.

Theorem 3.7 $\frac{1}{np}\sum X^{q}$ is a minimum variance bound (MVB) estimator for the scale parameter θ of the class of life-time distributions.

Proof: Let $\underset{-}{X}=(X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from a population havingp.d.f (1) which satisfies (2) and (3). Then the likelihood function and log-likelihood function of θ given $\underset{-}{X}$ in (9) and (10) are as follows;

$$L(\theta \mid \underset{-}{X}) = \exp[b(\theta\sum a(X) + nc(\theta) + \sum h(X))]$$

$$\ln L(\theta \mid \underset{-}{X}) = b(\theta)\sum a(X) + nc(\theta) + \sum h(X),$$

where $a(X) = X^{q}$, $b(\theta) = -\frac{1}{\theta}$, $c(\theta) = -p\log\theta$ and

$h(X) = \log\left(\frac{q}{\Gamma p} X^{pq-1}\right)$.

Now, differentiating the log-likelihood function with respect to θ we get

$$\frac{\delta \ln L(\theta \mid \underset{-}{X})}{\delta\theta} = \frac{\frac{1}{np}\sum a(X) - \theta}{\theta^2 / np} = \frac{\frac{1}{np}\sum X^{q} - \theta}{\theta^2 / np}. \qquad (11)$$

Therefore, $\frac{1}{np}\sum X^{q}$ is the minimum variance bound (MVB) estimator for the scale parameter θ of the class of life-time distributions.

Theorem 3.8 $\frac{1}{np}\sum X^{q}$ is the uniformly minimum variance unbiased estimator (UMVUE) for the scale parameter θ of the class of life-time distributions with minimum variance (MV) $\frac{\theta^2}{np}$.

Proof: Suppose $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from a population having p.d.f (1), satisfies (2) and (3). Then the likelihood function in (9) is

$$L(\theta \mid \underline{X}) = \exp\left[b(\theta\sum a(X) + nc(\theta) + \sum h(X))\right],$$

where $a(X) = X^q$, $b(\theta) = -\frac{1}{\theta}$, $c(\theta) = -p\log\theta$ and

$h(X) = \log\left(\frac{q}{\Gamma p} X^{pq-1}\right)$.

In above Theorem 3.4 and Theorem 3.5, it has been seen that the statistic $\sum a(X) = \sum X^q$ is a sufficient and a complete minimal sufficient statistic for θ. Hence according to the Lehmann-Scheffe [1] Theorem , $\frac{1}{n}\sum a(X)$ is the uniformly minimum variance unbiased estimator (UMVUE) of its expected value which is a function of parameter θ.

As $-2b(\theta)a(X)$ follows central Chi-square distribution with k degrees of freedom, therefore

$E[-2b(\theta)a(X)] = k$ and $V[-2b(\theta)a(X)] = 2k$ where $k = 2p$, $p > 0$.

$$\text{Thus, } E\left[\frac{\sum a(X)}{n}\right] = -\frac{k}{2b(\theta)}, \text{ it gives } E\left[\frac{1}{np}\sum X^q\right] = \theta,$$

$$\text{and } V\left[\frac{\sum a(X)}{n}\right] = \frac{k}{2nb^2(\theta)}, \text{ it also gives } V\left[\frac{1}{np}\sum X^q\right] = \frac{\theta^2}{np}.$$

Therefore, $\frac{1}{np}\sum X^q$ is the uniformly minimum variance unbiased estimator (UMVUE) for the scale parameter θ of the class of life-time distributions with minimum variance (MV) $\frac{\theta^2}{np}$.

Theorem 3.9 $\frac{1}{(np+1)}\sum X^q$ is a Pitman estimator for the scale parameter θ of the class of life-time distributions.

Proof: Suppose X be a random variable which follows the class of life-time distributions (1), satisfies (2) and (3). Let us transform $-b(\theta)a(X)=\frac{1}{\theta}x^q=\frac{1}{2}z$ in (1), it gives

$$f(z)=\frac{1}{2^p\Gamma p}e^{-\frac{1}{2}z}z^{p-1};\quad z\geq 0,\ p>0, \tag{12}$$

which is distributed as central Chi-square with p degrees of freedom and also independent of θ, hence θ is a scale parameter.

Let $\underset{-}{X}=(X_1,X_2,\cdots,X_n)$ be a random sample of size n drawn from a population having p.d.f (1). The Pitman [8] estimator for the scale parameter θ is defined as

$$T=t(X_1,X_2,\cdots,X_n)=\frac{\int_0^{\infty}\frac{1}{\theta^2}\prod_{i=1}^{n}f(x_i;\theta)d\theta}{\int_0^{\infty}\frac{1}{\theta^3}\prod_{i=1}^{n}f(x_i;\theta)d\theta}$$

$$=\frac{\int_0^{\infty}\left(\frac{1}{\theta}\right)^{np+2}e^{-\frac{1}{\theta}\sum x^q}d\theta}{\int_0^{\infty}\left(\frac{1}{\theta}\right)^{np+3}e^{-\frac{1}{\theta}\sum x^q}d\theta}. \tag{13}$$

Using the transformation $\frac{1}{\theta}\sum x^q=\frac{1}{2}z$ in (13), it gives

$$T = t(X_1, X_2, \cdots, X_n) = \frac{1}{(np+1)} \sum x^q ,$$

which is a biased estimator for θ and also function of a sufficient statistic using the Theorem 3.4.

Therefore, $\frac{1}{(np+1)} \sum X^q$ is a Pitman estimator for the scale parameter θ of the class of life-time distributions.

Theorem 3.10 $\left[\frac{2\sum X^q}{\chi^2_{2np,\,(1-\alpha)}}, \quad \frac{2\sum X^q}{\chi^2_{2np,\,\alpha}} \right]$ is a $100(1-(\alpha_1+\alpha_2))$ confidence interval for the scale Parameter θ of the class of life-time distributions.

Proof: Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from the population having p.d.f

(1), satisfies (2) and (3). Then the likelihood function in (9) as

$$L(\theta \mid \underline{X}) = \exp\left[b(\theta)\sum a(X) + nc(\theta) + \sum h(X)\right],$$

where $a(X) = X^q$, $b(\theta) = -\frac{1}{\theta}$, $c(\theta) = -p\log\theta$ and

$$h(X) = \log\left(\frac{q}{\Gamma p} X^{pq-1}\right).$$

Using Theorem 3.3(c), we consider $-2b(\theta)\sum a(X) = \frac{2}{\theta}\sum X_i^{\,q}$ as a central Chi-square variatewith $2np$, $p > 0$ degrees of freedom and a pivot, the distribution of which is independent of θ.

Thus

$$P\left[\chi^2_{2np,\alpha} < \frac{2\sum X^q}{\theta} < \chi^2_{2np,(1-\alpha)}\right] = 1-\alpha \text{ , where } \alpha_1 + \alpha_2 = \alpha$$

$$\Rightarrow P\left[\frac{2\sum X^q}{\chi^2_{2np,(1-\alpha)}} < \theta < \frac{2\sum X^q}{\chi^2_{2np,\ \alpha_1}}\right] = 1-\alpha\,.$$

Hence, $\left[\frac{2\sum X^q}{\chi^2_{2np,(1-\alpha)}}, \quad \frac{2\sum X^q}{\chi^2_{2np,\ \alpha}}\right]$ is a $100(1-(\alpha_1+\alpha_2))$ confidence interval for the scale parameter θ of the class of life-time distributions.

Theorem 3.11 The class of life-time distributions family has monotonic likelihood ratio (MLR).

Proof: Let $\underset{-}{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from a population having p.d.f (1), satisfies (2) and (3). Then the likelihood function in (9) as

$$L(\theta \mid \underset{-}{X}) = \exp\left[b(\theta)\sum a(X) + nc(\theta) + \sum h(X)\right],$$

where $a(X) = X^q$, $b(\theta) = -\frac{1}{\theta}$, $c(\theta) = -p\log\theta$ and $h(X) = \log\left(\frac{q}{\Gamma p} X^{pq-1}\right)$.

Now consider the likelihood ratio for $\theta_1 < \theta_2$,

$$\frac{L(\theta_2 \mid \underset{-}{X})}{L(\theta_1 \mid \underset{-}{X})} = \exp\left[\frac{c(\theta_2)}{c(\theta_1)} + \{b(\theta_2) - b(\theta_1)\}\sum a(X)\right],$$

which is a non-decreasing in $\sum a(X)$ if $\{b(\theta_2) - b(\theta_1)\}$ is non-decreasing. Therefore, using [9] it has been seen that the class of life-time distributions family has monotonic likelihood ratio in $\sum a(X) = \sum X^q$.

Theorem 3.12 Suppose $\underset{-}{X} = (X_1, X_2, \cdots, X_n)$be a random sample of size n drawn from the class of life-time distributions family, then an α level one-sided uniformly most powerful(UMP) test for testing the hypothesis $H_0 : \theta \le \theta_0$ against $H_1 : \theta > \theta_0$ is $\varphi(\underset{-}{X}) = 1$ if

$\sum X^q > \frac{2}{\theta_0}\chi^2_{2np,\,(1-\alpha)}$, provided $b(\theta) = -\frac{1}{\theta}$ is strictly increasing in θ. Also the

power function of the test is $P_\theta\left[\chi^2_{2np} > \frac{\theta_0}{\theta}\chi^2_{2np,\,(1-\alpha)}\right]$.

Proof: Let $\underset{-}{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n drawn from a population having p.d.f (1), satisfies (2) and (3). Then the likelihood function in (9) as

$$L(\theta \mid \underset{-}{X}) = \exp\left[b(\theta)\sum a(X) + nc(\theta) + \sum h(X)\right],$$

where $a(X) = X^q$, $b(\theta) = -\frac{1}{\theta}$, $c(\theta) = -p\log\theta$ and

$$h(X) = \log\left(\frac{q}{\Gamma p}X^{pq-1}\right).$$

It was observed that in Theorem 3.11, the above distributions family (1) has a monotonic likelihood ratio. Let $b(\theta)$ be strictly increasing in θ and also let k be an integer. In case of $\sum a(X) = \sum X^q$, $b(\theta) = -\frac{1}{\theta}$ and $k = 2p$, $p > 0$, according to Karlin and Rubin [9], the one-sided UMP test of an α level for testing the null hypothesis $H_0: \theta \le \theta_0$ against alternative hypothesis $H_1: \theta > \theta_0$, the test function is defined as

$$\varphi(\underset{-}{X}) = \begin{cases} 1 & if \ \sum a(X) > c_0 \\ 0 & if \ \sum a(X) \le c_0 \end{cases},$$

where c_0 is a constant and obtained by the condition

$$P_{\theta_0}\left[\varphi(\underset{-}{X}) > c_0\right] = \alpha,$$

$$\text{or, } P_{\theta_0}\left[\sum a(X) > c_0\right] = \alpha$$

$$\text{or, } P_{\theta_0}\left[-2b(\theta_0)\sum a(X) > -2b(\theta_0)c_0\right] = \alpha$$

$$\text{or, } P_{\theta_0}\left[\frac{2}{\theta_0}\sum x^q > \frac{2c_0}{\theta_0}\right] = \alpha$$

$$\text{or } P_{\theta_0}\left[\chi^2_{2np} > \frac{2c_0}{\theta_0}\right] = \alpha .$$

Let $\frac{2c_0}{\theta_0} = \chi^2_{2np,\,(1-\alpha)}$, thus $c_0 = \frac{\theta_0}{2}\chi^2_{2np,\,(1-\alpha)}$ and the power function of the test will be defined by

$$P_\theta\left[\varphi(\underset{-}{X}) > c_0\right]$$

$$\text{or, } P_\theta\left[\sum a(X) > c_0\right]$$

$$\text{or, } P_\theta\left[\frac{2}{\theta}\sum x^q > \frac{2c_0}{\theta}\right]$$

$$\text{or, } P_\theta\left[\chi^2_{2np} > \frac{\theta_0}{\theta}\chi^2_{2np,\,(1-\alpha)}\right].$$

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